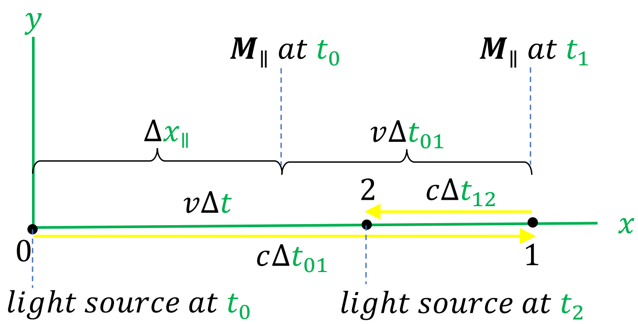
Special Relativity Length Contraction

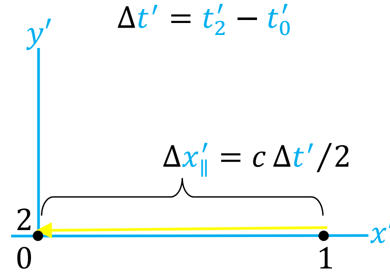
Derivation of the length contraction formula from the time dilation formula,
using the light path reflected off mirror $\boldsymbol{M}_\parallel$
which is offset from the light source parallel to the direction of motion
$(example\ has\ v = c/2)$

$event\ 0{:}\ send\ light\ pulse$
$event\ 1{:}\ bounce\ at\ mirror\ \boldsymbol{M}_\parallel\ (moving\ along\ x\ axis)$
$event\ 2{:}\ receive\ light\ pulse\ (also\ send\ next\ light\ pulse)$

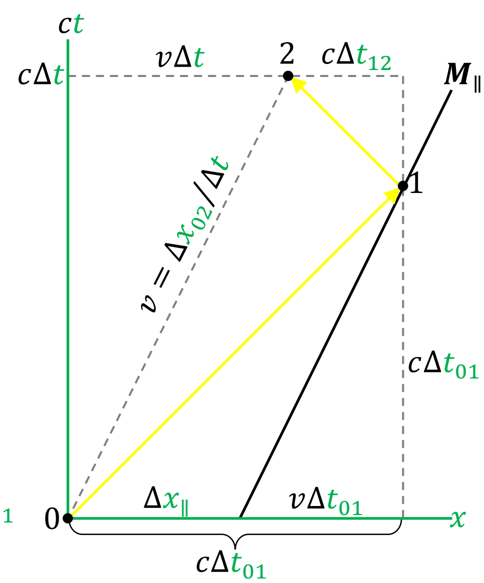
$\Delta t = \Delta t_{01} + \Delta t_{12} = \gamma \Delta t'$



$\Delta t'/2 = t_2' - t_1' = t_1' - t_0'$
$\Delta t' = t_2' - t_0'$



$mirror\ \boldsymbol{M}_\parallel\ is\ ahead\ of\ event\ 0\ in\ the\ direction\ of\ motion$
$S'\ moves\ at\ speed\ v\ in\ the\ +x\ direction\ relative\ to\ S$
$let\ \Delta t'\ be\ S'\ time\ from\ event\ 0\ to\ event\ 2 = one\ S'\ clock\ tick$
$let\ \Delta x_\parallel'\ be\ S'\ (proper)\ distance\ from\ event\ 0\ to\ \boldsymbol{M}_\parallel = c\,\Delta t'/2$
$let\ \Delta x_\parallel\ be\ instantaneous\ S\ distance\ from\ event\ 0\ to\ \boldsymbol{M}_\parallel$
$let\ \Delta t\ be\ S\ time\ from\ event\ 0\ to\ event\ 2 = one\ S\ clock\ tick$
$let\ \Delta t_{01}\ be\ S\ time\ from\ event\ 0\ to\ event\ 1$
$let\ \Delta t_{12}\ be\ S\ time\ from\ event\ 1\ to\ event\ 2$



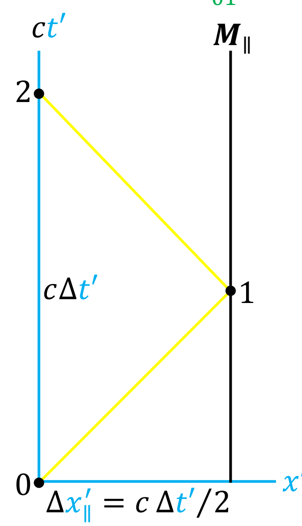
$S\ distance\ from\ event\ 0\ to\ event\ 1 = c\Delta t_{01} = \Delta x_\parallel + v\Delta t_{01}$
$\Delta x_\parallel = \Delta t_{01}(c - v) \qquad \Delta t_{01} = \Delta x_\parallel/(c - v) \qquad \Delta t_{12} = \Delta t - \Delta t_{01}$

$S\ distance\ from\ event\ 0\ to\ event\ 2 = v\Delta t = c\Delta t_{01} - c\Delta t_{12}$
$v\Delta t = c\Delta t_{01} - c(\Delta t - \Delta t_{01}) = 2c\Delta t_{01} - c\Delta t$
$\Delta t(c + v) = 2c\Delta t_{01} = 2c\,\Delta x_\parallel/(c - v)$
$\Delta x_\parallel = \Delta t(c + v)(c - v)/(2c) = \Delta t(c^2 - v^2)\,c/(2c^2)$

$use\ time\ dilation\ formula{:}\ \Delta t = \gamma \Delta t'$
$\Delta x_\parallel = \gamma \Delta t'(c^2 - v^2)\,c/(2c^2)$
$\quad = \gamma(c\Delta t'/2)(1 - v^2/c^2) = \gamma \Delta x_\parallel' \gamma^{-2} = \Delta x_\parallel'/\gamma$

$Generalize\ to\ any\ \Delta x'\ length\ moving\ relative\ to\ S{:}$

$$\boxed{\Delta x = \Delta x'/\gamma}$$



$\gamma \geq 1 \Rightarrow length\ contraction\ as\ seen\ by\ S\ frame{:}$
$S\ length\ \Delta x\ is\ less\ than\ S'\ length\ \Delta x'\ (by\ a\ factor\ of\ \gamma\ in\ the\ direction\ of\ motion)$
$S\ sees\ S'\ rulers\ to\ be\ shorter\ than\ S\ rulers\ (by\ a\ factor\ of\ \gamma\ in\ the\ direction\ of\ motion)$